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# Estimation of the spectral density of a homogeneous random stable discrete time field

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## Abstract

In earlier papers,  $2\pi$ -periodic spectral data windows have been used in spectral estimation of discrete-time random fields having finite second-order moments. In this paper, we show that  $2\pi$ -periodic spectral windows can also be used to construct estimates of the spectral density of a homogeneous symmetric  $\alpha$ -stable discrete-time random field. These fields do not have second-order moments if  $0 < \alpha < 2$ . We construct an estimate of the spectrum, calculate the asymptotic mean and variance, and prove weak consistency of our estimate.

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## 1 Introduction

The use of methods of spectral analysis of stochastic processes and random fields in many areas of scientific research has been considerably extended. Particular attention has been focused on methods of spectral analysis of continuous-time and discrete-time stationary processes and homogeneous fields.

As a consequence of this research the topic of spectral analysis of processes and fields having finite second-order moments is better represented in the literature. The study of random processes and fields having no second-order moments is therefore of particular importance. In  $\alpha$ -stable stochastic processes and random fields,  $0 < \alpha \leq 2$ ,

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only Gaussian processes and fields ( $\alpha = 2$ ) have finite variances. In the case where  $0 < \alpha < 2$ , there exist finite moments of order  $p \in (0, \alpha)$  only; see, e.g., section 2.1 in Zolotarev (1986). Therefore traditional methods cannot be applied for solving practical problems and the development of a special theory is required.

The research in multidimensional stable distributions required to develop a theory of random processes and fields (Paulauskas (1976), Press (1972)). A further development of this theory was influenced by the publication of results which are related to the representation of characteristic functions of multidimensional symmetric stable distributions and to the spectral representation of symmetric stable processes and fields. The notion of spectral density of a stable process introduced by Masry and Cambanis (1984) and the construction of estimates of the spectral density of a continuous-time stable random process are especially important for further research.

Different estimates of the spectral density of a homogeneous symmetric  $\alpha$ -stable discrete-time random field have been introduced and studied in many articles: Hosoya (1978), Masry and Cambanis (1984), Sabre (1995), Sabre (2000), to mention a few. However, either one-dimensional processes (Masry and Cambanis (1984)) or two-dimensional fields (Sabre (2000)) have been studied in these articles. The difficulty of studying stable fields consists in the fact that these fields do not have finite second-order moment either and, if  $0 < \alpha \leq 1$ , they do not have finite first-order moments either. In the literature (see, e.g., Sabre (1995)), mainly spectral windows usually used for continuous-time fields are considered. The problem of development of the corresponding technique for the case of discrete-time is less investigated. In this paper, we construct and study an estimate of the spectral density of a homogeneous symmetric  $\alpha$ -stable ( $0 < \alpha < 2$ ) discrete-time random field by means of  $2\pi$ -periodical spectral windows used for fields having finite second-order moments (Leonenko and Ivanov (1989)).

## 2 Assumptions and notations

Let us give our basic notations to be used in the article. Denote by  $\mathbb{N} = \{1, 2, \dots\}$  the set of natural numbers,  $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$  the set of integer numbers,  $\Pi^n = [-\pi, \pi]^n$ ,  $P^n = [-\tau_1, \tau_1] \times [-\tau_2, \tau_2] \times \dots \times [-\tau_n, \tau_n]$  an integer lattice of  $n$ -dimensional parallelepiped where  $\tau_j \in \mathbb{N}$ ,  $j = \overline{1, n}$ .  $T = (T_1, \dots, T_n)$  the  $n$ -dimensional vector having  $T_j = 2\tau_j + 1$ ,  $j = \overline{1, n}$ ;  $N_T = T_1 \times \dots \times T_n$ ;  $\langle a, b \rangle$  the inner product of the vectors  $a = (a_1, \dots, a_n)$  and  $b = (b_1, \dots, b_n)$ . The notation  $T \rightarrow \infty$  means that  $T_j \rightarrow \infty$ ,  $j = \overline{1, n}$ , we write  $\|a\| = \langle a, a \rangle^{\frac{1}{2}}$  and the notation  $x_T \cong y_T$  means that  $x_T - y_T \rightarrow 0$  as  $T \rightarrow \infty$ .

Denote by  $D(p) = \int_{-\infty}^{\infty} |u|^{-1-p} (1 - \cos(u)) du$ ,  $F(p, \alpha) = \int_{-\infty}^{\infty} |u|^{-1-p} (1 - e^{-|u|^\alpha}) du$ ,  $c_\alpha = \frac{1}{\alpha\pi} \int_0^\pi |\cos(u)|^\alpha du$ , and

$$k(p, \alpha) = \frac{D(p)}{F(p, \alpha) (c_\alpha)^{\frac{p}{\alpha}}}. \quad (1)$$

Let  $h_T(t)$ ,  $t = (\frac{t_1}{\tau_1}, \frac{t_2}{\tau_2}, \dots, \frac{t_n}{\tau_n})$ , be an  $n$ -dimensional data window,

$$H^{(T)}(\lambda) = \sum_{t \in P^n} h_T(t) \exp(-i \langle t, \lambda \rangle), \quad (2)$$

$$A_T = \left[ \int_{\Pi^n} |H^{(T)}(\lambda)|^\alpha d\lambda \right]^{-\frac{1}{\alpha}}. \quad (3)$$

Let us consider numerical sequences  $M_{T_j} \in \mathbb{N}$  and  $L_{T_j} \in \mathbb{N}$ ,  $j = \overline{1, n}$ , where  $M_{T_j} \rightarrow \infty$  as  $T_j \rightarrow \infty$  but  $\frac{M_{T_j}}{T_j} \rightarrow 0$  as  $T_j \rightarrow \infty$ ,  $j = \overline{1, n}$ ;  $L_{T_j} \rightarrow \infty$ ,  $\frac{M_{T_j}}{L_{T_j}} \rightarrow 0$ ,  $\frac{L_{T_j}}{T_j} \rightarrow 0$  as  $T_j \rightarrow \infty$  for all  $j = \overline{1, n}$ . Denote by

$$M_T = \prod_{j=1}^n M_{T_j} \quad (4)$$

$$L_T = \prod_{j=1}^n L_{T_j} \quad (5)$$

Let  $w_T(l) = w\left(\frac{l_1}{M_{T_1}}, \frac{l_2}{M_{T_2}}, \dots, \frac{l_n}{M_{T_n}}\right)$  be a  $n$ -dimensional correlation window,  $l = (l_1, l_2, \dots, l_n)$ , where  $w(x)$ ,  $x \in \mathbb{R}^n$ , satisfies the following conditions:

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} w(x) &= w(0) = 1, \\ 0 &\leq w(x) \leq 1, x \in \mathbb{R}^n, \\ \int_{\mathbb{R}^n} w^2(x) dx &< \infty, \end{aligned} \quad (6)$$

and  $W_T(\nu)$ ,  $\nu \in \Pi^n$ , is a nonnegative spectral data window of the form

$$W_T(\nu) = \frac{1}{(2\pi)^n} \sum_{l_1=-M_{T_1}}^{M_{T_1}} \sum_{l_2=-M_{T_2}}^{M_{T_2}} \dots \sum_{l_n=-M_{T_n}}^{M_{T_n}} w_T(l) \exp(-i \langle \nu, l \rangle), \nu \in \Pi^n. \quad (7)$$

Examples of spectral data windows can be found in Corollary 2, Appendix 2 (Example 3). See also Section 4.1 in Brillinger (1975) and Section 4.4 in Trush (1999).

### 3 Main results

The definition of a symmetric  $\alpha$ -stable random field is due to Nolan (1988). If  $X$  is a  $\alpha$ -stable random variable,  $0 < \alpha \leq 2$ , then put

$$\|X\|_\alpha = [-\ln(E \exp \{iX\})]^{-\frac{1}{\alpha}}. \quad (8)$$

This is a norm in the space of symmetric  $\alpha$ -stable random variables. Of course,  $\|X\|_2^2 = \frac{\text{var}(X)}{2}$  in the Gaussian case. Therefore one can think of  $\|X\|_p^p$  to be a generalization of the variance. It is known (Nolan (1988)) that for any  $0 < p < \alpha$  there exists a  $C(p, \alpha)$  such that

$$E |X|^p = C(p, \alpha) \|X\|_\alpha^p$$

for every symmetric  $\alpha$ -stable random variable  $X$ .

Let  $0 < \alpha \leq 2$  and  $\mathbb{T} \subset \mathbb{Z}^n$ . A complex-valued random field  $X(t)$ ,  $t \in \mathbb{T}$ , is called  $\alpha$ -stable if for any  $m \geq 1$ ,  $a_j \in \mathbb{C}$ ,  $t^{(j)} \in \mathbb{T}$ ,  $j = \overline{1, m}$ , every linear combination  $\sum_{j=1}^m a_j X(t^{(j)})$  is a symmetric  $\alpha$ -stable random variable. Then by (8)

$$E \exp \left\{ i \sum_{j=1}^m a_j X(t^{(j)}) \right\} = \exp \left( - \left\| \sum_{j=1}^m a_j X(t^{(j)}) \right\|_\alpha^\alpha \right).$$

Therefore  $\left\| \sum_{j=1}^m a_j X(t^{(j)}) \right\|_\alpha^\alpha$  determines completely the distribution of  $(X(t^{(1)}), X(t^{(2)}), \dots, X(t^{(m)}))$ . Each  $\alpha$ -stable random field  $X(t)$ ,  $t \in \mathbb{T}$ , has a representation as a stochastic integral: there exists a measurable space  $(\Omega, F, P)$  and a collection  $\{f(t, \cdot) : t \in \mathbb{T}\} \subset L^\alpha(\Omega, F, P)$  such that  $X(t) = \int f(t, u) \zeta(du)$ . Then  $\zeta$  is the  $\alpha$ -stable random measure generated by  $P$ . A method for constructing  $\zeta$  is described in Hardin (1982).

Later we shall consider a special class of symmetric  $\alpha$ -stable fields which are called harmonizable random fields. Let  $X(t)$ ,  $t \in \mathbb{Z}^n$ , be a homogeneous symmetric  $\alpha$ -stable ( $0 < \alpha < 2$ ) discrete-time random field having the spectral representation of the form

$$X(t) = \int_{\Pi^n} \exp(i \langle t, \lambda \rangle) d\xi(\lambda)$$

where  $t = (t_1, t_2, \dots, t_n)$ ,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ , and where  $\xi(\lambda)$  is  $\alpha$ -stable random field with independent increments such that  $[E|d\xi(\lambda)|^p]^{-\frac{\alpha}{p}} = C(p, \alpha)\varphi(\lambda)d\lambda$  for  $p \in (0, \alpha)$ . The constant  $C(p, \alpha)$  depends on  $p$  and  $\alpha$  only, the function  $\varphi(\lambda)$ ,  $\lambda \in \Pi^n$ , is a nonnegative even  $2\pi$ -periodic function in each argument on  $\mathbb{R}^n$ . This function is called spectral density of  $X(t)$ ,  $t \in \mathbb{Z}^n$ . If  $\alpha = 2$ , the field  $X(t)$ ,  $t \in \mathbb{Z}^n$ , is Gaussian, the function  $\varphi(\lambda)$ ,  $\lambda \in \Pi^n$ , is the "usual" spectral density and a standard spectral analysis is then applied. If  $0 < \alpha < 2$ , the function  $\varphi(\lambda)$ ,  $\lambda \in \Pi^n$ , does not represent spectral density in the usual sense. However, it was shown in Nolan (1988) that for linear prediction and filtering the role of this function is quite similar to that played by the spectral density function in second-order fields.

Let

$$\{X(t_1, t_2, \dots, t_n), t = (t_1, t_2, \dots, t_n) \in P^n\} \quad (9)$$

be  $N_T$  observations of the field  $X(t), t \in \mathbb{Z}^n$  on  $P^n$ .

**Definition 1** *The statistic*

$$d_T(\lambda) = A_T \sum_{t \in P^n} \cos(\langle t, \lambda \rangle) h_T(t) X(t), \lambda \in \Pi^n, \quad (10)$$

is called the finite modified Fourier transform of observations (9) of the field  $X(t), t \in \mathbb{Z}^n$ , where  $h_T(t), t \in \mathbb{Z}^n$ , is an  $n$ -dimensional data window and  $A_T$  is given by (3).

**Definition 2** *The statistic*

$$I_T(\lambda) = k(p, \alpha) |d_T(\lambda)|^p, \lambda \in \Pi^n, \quad (11)$$

is called the modified periodogram of observations (9) of the field  $X(t), t \in \mathbb{Z}^n$  where  $d_T(\lambda), \lambda \in \Pi^n$ , is given by (10) and  $k(p, \alpha)$  is defined by (1).

As an estimate of the function  $[\varphi(\lambda)]^{\frac{p}{\alpha}}, \lambda \in \Pi^n$ , we consider a smoothed periodogram of the form

$$\widehat{f}_T(\lambda) = \int_{\Pi^n} W_T(\nu) I_T(\lambda + \nu) d\nu, \lambda \in \Pi^n, \quad (12)$$

where  $\nu = (\nu_1, \nu_2, \dots, \nu_n), d\nu = d\nu_1 d\nu_2 \cdots d\nu_n$ , and where  $I_T(\lambda), \lambda \in \Pi^n$ , is given by (11) and the spectral window  $W_T(\nu), \nu \in \Pi^n$ , is defined by (7). Let us consider the statistic

$$\widehat{\varphi}_T(\lambda) = \left[ \widehat{f}_T(\lambda) \right]^{\frac{\alpha}{p}}, \lambda \in \Pi^n, \quad (13)$$

as an estimate of the spectral density  $\varphi(\lambda), \lambda \in \Pi^n$ .

**Definition 3** *The function  $\varphi(\lambda), \lambda \in \Pi^n$ , is said to satisfy the Hölder inequality at a point  $\lambda^{(0)} \in \Pi^n$  with index  $\gamma, 0 < \gamma \leq 1$ , if for any  $\lambda$  close enough to  $\lambda^{(0)}$ , the following inequality holds:*

$$\left| \varphi(\lambda) - \varphi(\lambda^{(0)}) \right| \leq A(\lambda^{(0)}) \|\lambda - \lambda^{(0)}\|^\gamma \quad (14)$$

where  $0 < A(\lambda^{(0)}) < +\infty$ .

**Definition 4** *A set of even  $2\pi$ -periodic in each argument functions  $G_T(\lambda), \lambda \in \Pi^n$ , is called a kernel on  $\Pi^n$  if it satisfies the following conditions:*

$$\int_{\Pi^n} G_T(\lambda) d\lambda = 1 \text{ for any } T = (t_1, t_2, \dots, t_n),$$

$$\int_{\Pi^n \setminus \{\|\lambda\| \leq \delta\}} G_T(\lambda) d\lambda \longrightarrow 0 \text{ for any fixed } \delta \in (0, \pi) \text{ as } T \rightarrow \infty.$$

We begin with considering the mathematical expectation and the variance of estimator (12) in general. Then we examine the use of specific data windows  $h_T(t)$  and spectral data windows  $w_T(l)$  and we prove convergence in probability of statistics (13). The proofs of all results in this section are given in Appendix 1.

**Theorem 1** Assume that the spectral density  $\varphi(\lambda)$ ,  $\lambda \in \Pi^n$ , of the field  $X(t)$ ,  $t \in \mathbb{Z}^n$ , is bounded on  $\Pi^n$ , positive and satisfies the Hölder inequality (14) at the point  $\lambda^{(0)} \in \Pi^n$ . Let  $|H_T(\lambda)|^\alpha = |A_T H^{(T)}(\lambda)|^\alpha$ ,  $\lambda \in \Pi^n$ , be a kernel on  $\Pi^n$  where  $A_T$  and  $H^{(T)}(\lambda)$ ,  $\lambda \in \Pi^n$  are defined by (3) and (2), respectively. Assume also that

$$\int_{\Pi^n \setminus \{\|\lambda\| \leq \delta\}} |H_T(\lambda)|^\alpha d\lambda = o\left(\int_{\Pi^n} \|\lambda\|^\gamma |H_T(\lambda)|^\alpha d\lambda\right) \text{ for any fixed } \delta \in (0, \pi) \text{ as } T \rightarrow \infty, \quad (15)$$

$$\int_{\Pi^n} \|\lambda\|^\gamma |H_T(\lambda)|^\alpha d\lambda \longrightarrow 0 \text{ as } T \rightarrow \infty, \quad (16)$$

$$\int_{\Pi^n \setminus \{\|v\| \leq \delta\}} W_T(v) dv = o\left(\int_{\Pi^n} \|v\|^\gamma W_T(v) dv\right) \text{ for any fixed } \delta \in (0, \pi) \text{ as } T \rightarrow \infty, \quad (17)$$

$$\int_{\Pi^n} \|v\|^\gamma W_T(v) dv \longrightarrow 0 \text{ as } T \rightarrow \infty. \quad (18)$$

Then

$$\begin{aligned} & \left| E \widehat{f_T}(\lambda^{(0)}) - [\varphi(\lambda^{(0)})]^{\frac{p}{\alpha}} \right| \leq \\ & \leq C_1 A(\lambda^{(0)}) \left( \int_{\Pi^n} \|v\|^\gamma W_T(v) dv (1 + o(1)) + \int_{\Pi^n} \|\mu\|^\gamma |H_T(\mu)|^\alpha d\mu (1 + o(1)) \right) \text{ as } T \rightarrow \infty \end{aligned} \quad (19)$$

where

$$C_1 = \frac{p}{2\alpha} \max_{v \in \Pi^n} \left| [\psi_T(\lambda^{(0)} + v)]^{\frac{p}{\alpha}-1} + [\varphi(\lambda^{(0)})]^{\frac{p}{\alpha}-1} \right|, \quad (20)$$

$$\psi_T(\lambda) = \int_{\Pi^n} |H_T(v)|^\alpha \varphi(\lambda + v) dv, \lambda \in \Pi^n. \quad (21)$$

Examples of the data windows satisfying conditions (15) and (16) and also spectral data windows satisfying conditions (17) and (18) can be found in Appendix 2 (Examples 1 and 3, respectively).

**Corollary 1** Assume that the correlation window  $w_T(l)$  can be written as

$$w_T(l) = \prod_{j=1}^n w_{T_j}(l_j), \quad (22)$$

where  $w_{T_j}(l_j) = w_j\left(\frac{l_j}{M_{T_j}}\right)$ ,  $j = \overline{1, n}$ , are even functions satisfying the following conditions:

$$\begin{aligned} \sup_{x \in \mathbb{R}} w_j(x) &= w_j(0) = 1, \\ 0 &\leq w_j(x) \leq 1, x \in \mathbb{R}, \\ \int_{-\infty}^{+\infty} w_j^2(x) dx &< \infty, j = \overline{1, n}. \end{aligned}$$

Suppose that the function

$$H_T(\lambda) = \prod_{j=1}^n H_{T_j}(\lambda_j) \quad (23)$$

where  $|H_{T_j}(\lambda_j)|^\alpha = |A_{T_j} H^{(T_j)}(\lambda_j)|^\alpha$  are kernels on  $\Pi$ ,  $A_{T_j} = \left[ \int_{\Pi} |H^{(T_j)}(\lambda_j)|^\alpha d\lambda_j \right]^{-\frac{1}{\alpha}}$ ,  $H^{(T_j)}(\lambda_j) = \sum_{t_j=-\tau_j}^{\tau_j} \exp(-it_j \lambda_j) h_j\left(\frac{t_j}{\tau_j}\right)$ ,  $h_j\left(\frac{t_j}{\tau_j}\right)$  are one-dimensional data windows  $j = \overline{1, n}$ . Then

$$\begin{aligned} &\left| E \widehat{f_T}(\lambda^{(0)}) - [\varphi(\lambda^{(0)})]^\alpha \right| \leq \\ &\leq C_2 \left( \max_{j=\overline{1, n}} \int_{\Pi} |\nu_j|^\gamma W_{T_j}(\nu_j) d\nu_j (1 + o(1)) + \max_{j=\overline{1, n}} \int_{\Pi} |\mu_j|^\gamma |H_{T_j}(\mu_j)|^\alpha d\mu_j (1 + o(1)) \right) \\ &\text{as } T \rightarrow \infty \end{aligned}$$

where

$$C_2 = 2^n n C_1 A(\lambda^{(0)}). \quad (24)$$

**Corollary 2** Let the coordinate  $j$  of the number of observations  $N_T$  be given in the form  $T_j = 2k_j(m_j - 1) + 1$ ,  $k_j \in \left\{\frac{1}{2}\right\} \cup \mathbb{N}$ ,  $m_j \in \mathbb{N}$ ,  $j = \overline{1, n}$ . Assume that the assumptions in Corollary 1 hold and the functions  $w_j(x) = \widetilde{w}(x)$ ,  $j = \overline{1, n}$ , are of the form

$$\widetilde{w}(x) = \begin{cases} 1 - |x|, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1, \end{cases} \quad (25)$$

and the functions  $h_j\left(\frac{t_j}{\tau_j}\right) = h(k_j, m_j)$  are polynomial data windows. These windows can be represented as

$$|H_{T_j}(\lambda_j)|^\alpha = \frac{1}{B_{T_j, \alpha}} \left| \frac{\sin\left(\frac{m_j \lambda_j}{2}\right)}{\sin\left(\frac{\lambda_j}{2}\right)} \right|^{2k_j \alpha}, \lambda_j \in \Pi, \quad (26)$$

(see Section 4.4 in Trush (1999) and Demesh (1988)) where

$$B_{T_j, \alpha} = \int_{-\pi}^{\pi} \left| \frac{\sin\left(\frac{m_j \lambda_j}{2}\right)}{\sin\left(\frac{\lambda_j}{2}\right)} \right|^{2k_j \alpha} d\lambda_j, j = \overline{1, n}.$$

Then

$$\left| E\widehat{f}_T(\lambda^{(0)}) - [\varphi(\lambda^{(0)})]^\frac{p}{\alpha} \right| \leq C_2 C_3 \begin{cases} \max_{j=\overline{1,n}} \frac{1}{M_{T_j}^\gamma}, & \text{if } 0 < \gamma < 1, \\ \max_{j=\overline{1,n}} \frac{\ln(M_{T_j})}{M_{T_j}}, & \text{if } \gamma = 1, \end{cases} \quad (27)$$

for  $k_j > \frac{1+\gamma}{2\alpha}$ ,  $j = \overline{1,n}$ , where  $C_2$  is defined by (24) and where

$$C_3 = \begin{cases} \frac{2\gamma\pi}{(1-\gamma^2)}, & \text{if } 0 < \gamma < 1, \\ 2\pi, & \text{if } \gamma = 1. \end{cases}$$

If  $k_j = \frac{1}{2}$ , then the function

$$h\left(\frac{t_j}{\tau_j}\right) = \begin{cases} 1, & \text{if } |t_j| \leq \tau_j, \\ 0, & \text{if } |t_j| > \tau_j, \end{cases}$$

is the unit data window,  $j = \overline{1,n}$ . If  $k_j = 1$ , then the function

$$h\left(\frac{t_j}{\tau_j}\right) = \begin{cases} 1 - |t_j|, & \text{if } |t_j| \leq \tau_j, \\ 0, & \text{if } |t_j| > \tau_j, \end{cases}$$

is the triangle data window,  $j = \overline{1,n}$ .

**Theorem 2** Assume that the spectral density  $\varphi(\lambda)$ ,  $\lambda \in \Pi^n$ , of the field  $X(t)$ ,  $t \in \mathbb{Z}^n$  is bounded on  $\Pi^n$ , positive and continuous at the point  $\lambda^{(0)} \in \Pi^n$ . Suppose that the expression  $|H_T(\lambda)|^\alpha = |A_T H^{(T)}(\lambda)|^\alpha$ ,  $\lambda \in \Pi^n$ , is a kernel on  $\Pi^n$  where  $A_T$  and  $H^{(T)}(\lambda)$ ,  $\lambda \in \Pi^n$  are defined by (3) and (2), respectively, and let

$$\max_{s,r=\overline{1,L_T}, s \neq r} \int_{\Pi^n} \left| H_T(v^{(s)} - v) H_T(v^{(r)} - v) \right|^\frac{q}{2} dv \longrightarrow 0 \text{ as } T \rightarrow \infty. \quad (28)$$

Then

$$\text{var} \widehat{f}_T(\lambda^{(0)}) = O\left(\frac{M_T}{L_T}\right) + O\left(\int_{\Pi^n} \left| H_T(v^{(s)} - v) H_T(v^{(r)} - v) \right|^\frac{q}{2} dv\right) \quad (29)$$

where  $L_T$  and  $M_T$  are given in (5) and (4), respectively.

Examples of data windows satisfying condition (28) can be found in Appendix 2 (Example 2).

**Corollary 3** Assume that  $H_T(\lambda)$ ,  $\lambda \in \Pi^n$ , satisfies (23) and suppose that the assumptions in Theorem 2 hold. Then



$$\text{var} \widehat{f}_T(\lambda^{(0)}) = O\left(\frac{M_T}{L_T}\right) + O\left(\max_{j=\overline{1,n}, s, r=\overline{1, L_T}, s \neq r} \left( \int_{\Pi} \left| H_{T_j}(v_j^{(s)} - v_j) H_{T_j}(v_j^{(r)} - v_j) \right|^{\frac{\alpha}{2}} dv_j \right)\right). \quad (30)$$

**Corollary 4** Assume that  $H_{T_j}(\lambda_j)$ ,  $j = \overline{1, n}$ , satisfy (26) and suppose that the assumptions in Corollary 3 hold. Let also  $k_j = k$ ,  $j = \overline{1, n}$ . Then

$$\text{var} \widehat{f}_T(\lambda^{(0)}) = O\left(m^{-(\beta-q)n}\right) + O\left(m^{-\frac{2k^2\alpha^2(1-\beta)-1}{1+2k\alpha}}\right) \quad (31)$$

where  $M_{T_j} = m^q$ ,  $L_{T_j} = m^\beta$ ,  $0 < q < \beta < 1$ ,  $j = \overline{1, n}$ .

For the data windows in Corollary 2 and the spectral data windows described in Corollary 4, we obtain the rate of convergence in probability.

**Theorem 3** Suppose that the assumptions in Theorem 1 and Theorem 2 hold. Then for any  $\epsilon > 0$

$$P\left(\left|\widehat{\varphi}_T(\lambda^{(0)}) - \varphi(\lambda^{(0)})\right| > \epsilon\right) \longrightarrow 0 \text{ as } T \longrightarrow \infty$$

where  $\widehat{\varphi}_T(\lambda)$ ,  $\lambda \in \Pi^n$ , is given by (13).

**Corollary 5** Suppose that the assumptions in Corollary 2 and Corollary 4 hold. Then

$$a_m \left| \widehat{\varphi}(\lambda^{(0)}) - \varphi(\lambda^{(0)}) \right| \xrightarrow{P} 0$$

where

$$a_m = \begin{cases} m^{\frac{1}{2} \frac{\gamma n(2k^2\alpha^2-1)}{n\gamma(1+2k\alpha)+k^2\alpha^2(n+2\gamma)}} \frac{1}{\ln(m)} & \text{if } 0 < \gamma < 1, \\ m^{\frac{1}{2} \frac{n(2k^2\alpha^2-1)}{n(1+2k\alpha)+k^2\alpha^2(n+2)}} \frac{1}{\ln^2(m)} & \text{if } \gamma = 1, \end{cases}$$

when  $2k^2\alpha^2 > 1$ ,  $2k\alpha > 1 + \gamma$ .

## 4 Appendix 1

*Proof of Theorem 1.* As it follows from Trush and Orlova (1994), we have  $E I_T(\lambda) = [\psi_T(\lambda)]^{\frac{p}{\alpha}}$ ,  $\lambda \in \Pi^n$ . Then

$$\begin{aligned} \left| E \widehat{f}_T(\lambda^{(0)}) - [\varphi(\lambda^{(0)})]^{\frac{p}{\alpha}} \right| &= \left| E \int_{\Pi^n} W_T(v) I_T(\lambda^{(0)} + v) dv - [\varphi(\lambda^{(0)})]^{\frac{p}{\alpha}} \right| = \\ &= \left| \int_{\Pi^n} W_T(v) [\psi_T(\lambda^{(0)} + v)]^{\frac{p}{\alpha}} dv - [\varphi(\lambda^{(0)})]^{\frac{p}{\alpha}} \right| = \end{aligned}$$

$$\begin{aligned}
&= \left| \int_{\Pi^n} W_T(\nu) \left[ \left[ \psi_T(\lambda^{(0)} + \nu) \right]^{\frac{p}{\alpha}} - \left[ \varphi(\lambda^{(0)}) \right]^{\frac{p}{\alpha}} \right] d\nu \right| \leq \\
&\leq \int_{\Pi^n} W_T(\nu) \left| \left[ \psi_T(\lambda^{(0)} + \nu) \right]^{\frac{p}{\alpha}} - \left[ \varphi(\lambda^{(0)}) \right]^{\frac{p}{\alpha}} \right| d\nu.
\end{aligned}$$

By the inequality

$$|a^q - b^q| \leq \frac{q}{2} |a - b| (a^{q-1} + b^{q-1}) \quad (32)$$

valid for all  $a, b > 0$ ,  $q \in (0, 1]$ ,  $q \geq 2$ , and assuming  $a = \psi_T(\lambda^{(0)} + \nu)$ ,  $b = \varphi(\lambda^{(0)})$ ,  $q = \frac{p}{\alpha}$ , we obtain

$$\begin{aligned}
&\int_{\Pi^n} W_T(\nu) \left| \left[ \psi_T(\lambda^{(0)} + \nu) \right]^{\frac{p}{\alpha}} - \left[ \varphi(\lambda^{(0)}) \right]^{\frac{p}{\alpha}} \right| d\nu \leq \\
&\leq \frac{p}{2\alpha} \max_{\nu \in \Pi^n} \left| \left[ \psi_T(\lambda^{(0)} + \nu) \right]^{\frac{p}{\alpha}-1} + \left[ \varphi(\lambda^{(0)}) \right]^{\frac{p}{\alpha}-1} \right| \times \\
&\quad \times \int_{\Pi^n} W_T(\nu) \left| \psi_T(\lambda^{(0)} + \nu) - \varphi(\lambda^{(0)}) \right| d\nu \leq \\
&\leq C_1 \int_{\Pi^n} W_T(\nu) \int_{\Pi^n} |H_T(\mu)|^\alpha \left| \varphi(\lambda^{(0)} + \nu + \mu) - \varphi(\lambda^{(0)}) \right| d\mu d\nu.
\end{aligned}$$

By (14), (15), (17) and (20), we have for any fixed  $\delta$ ,  $0 < \delta < \pi$ ,

$$\begin{aligned}
&C_1 \int_{\Pi^n} W_T(\nu) \int_{\Pi^n} |H_T(\mu)|^\alpha \left| \varphi(\lambda^{(0)} + \nu + \mu) - \varphi(\lambda^{(0)}) \right| d\mu d\nu = \\
&= C_1 \int_{\|\nu\| \leq \delta} W_T(\nu) \int_{\|\mu\| \leq \delta} |H_T(\mu)|^\alpha \left| \varphi(\lambda^{(0)} + \nu + \mu) - \varphi(\lambda^{(0)}) \right| d\mu d\nu + \\
&+ C_1 \int_{\Pi^n \setminus \{\|\nu\| \leq \delta\}} W_T(\nu) \int_{\|\mu\| \leq \delta} |H_T(\mu)|^\alpha \left| \varphi(\lambda^{(0)} + \nu + \mu) - \varphi(\lambda^{(0)}) \right| d\mu d\nu + \\
&+ C_1 \int_{\Pi^n} W_T(\nu) \int_{\Pi^n \setminus \{\|\mu\| \leq \delta\}} |H_T(\mu)|^\alpha \left| \varphi(\lambda^{(0)} + \nu + \mu) - \varphi(\lambda^{(0)}) \right| d\mu d\nu \leq \\
&\leq C_1 A(\lambda^{(0)}) \int_{\|\nu\| \leq \delta} W_T(\nu) \int_{\|\mu\| \leq \delta} |H_T(\mu)|^\alpha \|\nu + \mu\|^\gamma d\mu d\nu + \\
&+ C_1 \max_{\nu, \mu \in \Pi^n} \left| \varphi(\lambda^{(0)} + \nu + \mu) - \varphi(\lambda^{(0)}) \right| \int_{\Pi^n \setminus \{\|\mu\| \leq \delta\}} |H_T(\mu)|^\alpha d\mu + \\
&+ C_1 \max_{\nu, \mu \in \Pi^n} \left| \varphi(\lambda^{(0)} + \nu + \mu) - \varphi(\lambda^{(0)}) \right| \int_{\Pi^n \setminus \{\|\nu\| \leq \delta\}} W_T(\nu) d\nu \leq \\
&\leq C_1 A(\lambda^{(0)}) \int_{\Pi^n} \|\nu\|^\gamma W_T(\nu) d\nu (1 + o(1)) + C_1 A(\lambda^{(0)}) \int_{\Pi^n} \|\mu\|^\gamma |H_T(\mu)|^\alpha d\mu (1 + o(1)) \\
&\quad \text{as } T \rightarrow \infty.
\end{aligned}$$

Hence

$$\begin{aligned} & \left| E\widehat{f_T}(\lambda^{(0)}) - [\varphi(\lambda^{(0)})]^{\frac{\gamma}{\alpha}} \right| \leq \\ & \leq C_1 A(\lambda^{(0)}) \left( \int_{\Pi^n} \|\nu\|^\gamma W_T(\nu) d\nu (1 + o(1)) + \int_{\Pi^n} \|\mu\|^\gamma |H_T(\mu)|^\alpha d\mu (1 + o(1)) \right) \\ & \text{as } T \rightarrow \infty. \end{aligned}$$

Theorem 1 is proved.  $\square$

*Proof of Corollary 1.* By (14) and (22), we obtain  $W_T(\nu) = \prod_{j=1}^n W_{T_j}(\nu_j)$  where

$$W_{T_j}(\nu_j) = \sum_{l_j=-M_{T_j}}^{M_{T_j}} w_j\left(\frac{l_j}{M_{T_j}}\right) \exp(-i\nu_j l_j).$$

Then

$$\begin{aligned} \int_{\Pi^n} \|\nu\|^\gamma W_T(\nu) d\nu &= \int_{\Pi^n} W_{T_1}(\nu_1) \times W_{T_2}(\nu_2) \times \cdots \times W_{T_n}(\nu_n) \left( \sum_{j=1}^n \nu_j^2 \right)^{\frac{\gamma}{2}} d\nu_1 d\nu_2 \cdots d\nu_n \leq \\ &\leq \int_{\Pi^n} W_{T_1}(\nu_1) \times W_{T_2}(\nu_2) \times \cdots \times W_{T_n}(\nu_n) \left( 2^n \sum_{j=1}^n |\nu_j|^\gamma \right) d\nu_1 d\nu_2 \cdots d\nu_n = \\ &= 2^n \sum_{j=1}^n \int_{\Pi} |\nu_j|^\gamma W_{T_j}(\nu_j) d\nu_j \leq 2^n n \max_{j=1, n} \left( \int_{\Pi} |\nu_j|^\gamma W_{T_j}(\nu_j) d\nu_j \right). \end{aligned}$$

Therefore

$$\int_{\Pi^n} \|\nu\|^\gamma W_T(\nu) d\nu \leq 2^n n \max_{j=1, n} \left( \int_{\Pi} |\nu_j|^\gamma W_{T_j}(\nu_j) d\nu_j \right). \quad (33)$$

By (23), we obtain

$$\begin{aligned} & \int_{\Pi^n} \|\mu\|^\gamma |H_T(\mu)|^\alpha d\mu = \\ & \int_{\Pi^n} |H_{T_1}(\mu_1)|^\alpha \times |H_{T_2}(\mu_2)|^\alpha \times \cdots \times |H_{T_n}(\mu_n)|^\alpha \left( \sum_{j=1}^n \mu_j^2 \right)^{\frac{\gamma}{2}} d\mu_1 d\mu_2 \cdots d\mu_n \leq \\ & \leq \int_{\Pi^n} |H_{T_1}(\mu_1)|^\alpha \times |H_{T_2}(\mu_2)|^\alpha \times \cdots \times |H_{T_n}(\mu_n)|^\alpha \left( 2^n \sum_{j=1}^n |\mu_j|^\gamma \right) d\mu_1 d\mu_2 \cdots d\mu_n \leq \\ & \leq 2^n \sum_{j=1}^n \int_{\Pi} |\mu_j|^\gamma |H_{T_j}(\mu_j)|^\alpha d\mu_j \leq 2^n n \max_{j=1, n} \left( \int_{\Pi} |\mu_j|^\gamma |H_{T_j}(\mu_j)|^\alpha d\mu_j \right). \end{aligned}$$

Therefore

$$\int_{\Pi^n} \|\mu\|^\gamma |H_T(\mu)|^\alpha d\mu \leq 2^n n \max_{j=1,n} \left( \int_{\Pi} |\mu_j|^\gamma |H_{T_j}(\mu_j)|^\alpha d\mu_j \right). \quad (34)$$

By substituting (33) and (34) in (19), we have

$$\begin{aligned} & \left| \widehat{f_T}(\lambda^{(0)}) - [\varphi(\lambda^{(0)})]^\alpha \right| \leq \\ & \leq C_2 \left( \max_{j=1,n} \int_{\Pi} |\nu_j|^\gamma W_{T_j}(\nu_j) d\nu_j (1 + o(1)) + \max_{j=1,n} \int_{\Pi} |\mu_j|^\gamma |H_{T_j}(\mu_j)|^\alpha d\mu_j (1 + o(1)) \right) \\ & \text{as } T \rightarrow \infty. \end{aligned}$$

Corollary 1 is proved.  $\square$

*Proof of Corollary 2.* By (25), we obtain

$$W_{T_j}(\nu_j) = \frac{1}{2\pi M_{T_j}} \frac{\sin^2\left(\frac{M_{T_j}\nu_j}{2}\right)}{\sin^2\left(\frac{\nu_j}{2}\right)}, \nu_j \in \Pi. \quad (35)$$

Since

$$\int_{\Pi} |\nu_j|^\gamma W_{T_j}(\nu_j) d\nu_j \leq \begin{cases} \frac{2^\gamma \pi}{(1-\gamma^2)} \frac{1}{M_{T_j}^\gamma}, & \text{if } 0 < \gamma < 1, \\ 2\pi \frac{\ln(M_{T_j})}{M_{T_j}}, & \text{if } \gamma = 1, \end{cases} \quad (36)$$

$j = \overline{1, n}$  (see Trush (1999)). It follows from Demesh (1988) that  $\int_{\Pi} |\mu_j|^\gamma |H_{T_j}(\mu_j)|^\alpha d\mu_j \leq \frac{\pi^\gamma (2k_j\alpha-1)}{2k_j\alpha(\gamma+1)} \frac{1}{T_j^\gamma}$ ,  $k_j > \frac{1+\gamma}{2\alpha}$ . By Corollary 1, we can write

$$\begin{aligned} & \left| E\widehat{f_T}(\lambda^{(0)}) - [\varphi(\lambda^{(0)})]^\alpha \right| \leq \\ & \leq C_2 C_3 \begin{cases} \max_{j=1,n} \left( \frac{2^\gamma \pi}{(1-\gamma^2)} \frac{1}{M_{T_j}^\gamma} (1 + o(1)) + \frac{\pi^\gamma (2k_j\alpha-1)}{2k_j\alpha(\gamma+1)} \frac{1}{T_j^\gamma} (1 + o(1)) \right), & \text{if } 0 < \gamma < 1 \text{ as } T \rightarrow \infty \\ \max_{j=1,n} \left( 2\pi \frac{\ln(M_{T_j})}{M_{T_j}} (1 + o(1)) + \frac{\pi^\gamma (2k_j\alpha-1)}{2k_j\alpha(\gamma+1)} \frac{1}{T_j^\gamma} (1 + o(1)) \right), & \text{if } \gamma = 1 \text{ as } T \rightarrow \infty. \end{cases} \end{aligned}$$

Since  $\frac{M_{T_j}}{T_j} \rightarrow 0$  as  $T_j \rightarrow \infty$ , the convergence to zero of the right-hand side in the last expression will be determined by  $\max_{j=1,n} \frac{1}{M_{T_j}^\gamma}$ , if  $0 < \gamma < 1$ , and by  $\max_{j=1,n} \frac{\ln(M_{T_j})}{M_{T_j}}$ , if  $\gamma = 1$ . Corollary 2 is proved.  $\square$

*Proof of Theorem 2.* We divide the coordinate  $j$  of the parallelepiped  $\Pi^n$  into  $L_{T_j}$  equal parts. These partitionings generate the partitioning of the parallelepiped  $\Pi^n$  into  $L_T$  parts. Let  $Q_s$  be the parallelepiped with number  $s$  and let  $\nu^{(s)}$  is some point belonging to  $Q_s$ ,  $s = \overline{1, L_T}$ .

We have

$$\begin{aligned} \text{var} \widehat{f}_T(\lambda^{(0)}) &= \text{var} \left( \int_{\Pi^n} W_T(\nu) I_T(\lambda^{(0)} + \nu) d\nu \right) \cong \text{var} \left( \sum_{s=1}^{L_T} W_T(\nu^{(s)}) I_T(\lambda^{(0)} + \nu^{(s)}) \frac{(2\pi)^n}{L_T} \right) = \\ &= \sum_{s=1}^{L_T} \text{var} \left( W_T(\nu^{(s)}) I_T(\lambda^{(0)} + \nu^{(s)}) \frac{(2\pi)^n}{L_T} \right) \text{ as } T \rightarrow \infty. \end{aligned} \quad (37)$$

As it follows from Orlova (1993), we have

$$\text{var} I_T(\lambda) = V(p, \alpha) [\psi_T(\lambda)]^{\frac{2p}{\alpha}}, \lambda \in \Pi^n, \quad (38)$$

where  $V(p, \alpha) = \frac{(k(p, \alpha))^2}{k(2p, \alpha)} - 1$  and where  $\psi_T(\lambda)$ ,  $\lambda \in \Pi^n$ , is given by (21). By substituting (38) into (37), we can write

$$\begin{aligned} \text{var} \widehat{f}_T(\lambda^{(0)}) &\cong \left( \frac{(2\pi)^n}{L_T} \right)^2 \sum_{s=1}^{L_T} W_T^2(\nu^{(s)}) V(p, \alpha) [\psi_T(\lambda^{(0)} + \nu^{(s)})]^{\frac{2p}{\alpha}} + \\ &+ \left( \frac{(2\pi)^n}{L_T} \right)^2 \sum_{s=1}^{L_T} \sum_{r=1, s \neq r}^{L_T} W_T(\nu^{(s)}) W_T(\nu^{(r)}) \text{cov} \{ I_T(\lambda^{(0)} + \nu^{(s)}), I_T(\lambda^{(0)} + \nu^{(r)}) \} \leq \\ &\leq \frac{(2\pi)^n}{L_T} \int_{\Pi^n} W_T^2(\nu) V(p, \alpha) [\psi_T(\lambda^{(0)} + \nu)]^{\frac{2p}{\alpha}} d\nu + \\ &+ \max_{s, r = \overline{1, L_T}, s \neq r} \text{cov} \{ I_T(\lambda^{(0)} + \nu^{(s)}), I_T(\lambda^{(0)} + \nu^{(r)}) \} \left( \frac{(2\pi)^n}{L_T} \right)^2 \sum_{s=1}^{L_T} \sum_{r=1, s \neq r}^{L_T} W_T(\nu^{(s)}) W_T(\nu^{(r)}). \end{aligned}$$

Since

$$V(p, \alpha) [\psi_T(\lambda^{(0)} + \nu)]^{\frac{2p}{\alpha}} < \infty \text{ and } \frac{(2\pi)^n}{L_T} \int_{\Pi^n} W_T^2(\nu) d\nu \cong \frac{M_T}{L_T} \int_{\mathbb{R}^n} w_T^2(x) dx \text{ as } T \rightarrow \infty,$$

we have

$$\frac{(2\pi)^n}{L_T} \int_{\Pi^n} W_T^2(\nu) V(p, \alpha) [\psi_T(\lambda^{(0)} + \nu)]^{\frac{2p}{\alpha}} d\nu = O\left(\frac{M_T}{L_T}\right) \text{ as } T \rightarrow \infty. \quad (39)$$

by (6) and by the properties of the sequences  $L_{T_j}$  and  $M_{T_j}$ ,  $j = \overline{1, n}$ .

Consider

$$\begin{aligned} \left( \frac{(2\pi)^n}{L_T} \right)^2 \sum_{s=1}^{L_T} \sum_{r=1, s \neq r}^{L_T} W_T(v^{(s)}) W_T(v^{(r)}) &\leq \left( \frac{(2\pi)^n}{L_T} \right)^2 \sum_{s=1}^{L_T} \sum_{r=1}^{L_T} W_T(v^{(s)}) W_T(v^{(r)}) = \\ &= \left( \frac{(2\pi)^n}{L_T} \sum_{s=1}^{L_T} W_T(v^{(s)}) \right)^2 \cong \left( \int_{\Pi^n} W_T(v) dv \right)^2 = 1 \text{ as } T \rightarrow \infty. \end{aligned}$$

As it follows from Orlova (1993), we have

$$\text{cov} \{ I_T(\lambda^{(0)} + v^{(s)}), I_T(\lambda^{(0)} + v^{(r)}) \} = O \left( \int_{\Pi^n} |H_T(v^{(s)} - v) H_T(v^{(r)} - v)|^{\frac{\alpha}{2}} dv \right).$$

By (39), we then obtain

$$\text{var} \widehat{f}_T(\lambda^{(0)}) = O \left( \frac{M_T}{L_T} \right) + O \left( \int_{\Pi^n} |H_T(v^{(s)} - v) H_T(v^{(r)} - v)|^{\frac{\alpha}{2}} dv \right).$$

Theorem 2 is proved.  $\square$

*Proof of Corollary 3.* We have

$$\begin{aligned} \int_{\Pi^n} |H_T(v^{(s)} - v) H_T(v^{(r)} - v)|^{\frac{\alpha}{2}} dv &= \int_{\Pi^n} |H_{T_1}(v_1^{(s)} - v_1) H_T(v_1^{(r)} - v_1)|^{\frac{\alpha}{2}} \times \\ &\times |H_{T_2}(v_2^{(s)} - v_2) H_T(v_2^{(r)} - v_2)|^{\frac{\alpha}{2}} \times \cdots \times |H_{T_n}(v_n^{(s)} - v_n) H_T(v_n^{(r)} - v_n)|^{\frac{\alpha}{2}} dv_1 dv_2 \cdots dv_n. \end{aligned}$$

The integral of the form  $\int_{\Pi} |H_{T_j}(v_j^{(s)} - v_j) H_T(v_j^{(r)} - v_j)|^{\frac{\alpha}{2}} dv_j$ ,  $j = \overline{1, n}$  is equal to 1 if  $v_j^{(s)} = v_j^{(r)}$ . Then

$$\begin{aligned} \int_{\Pi^n} |H_T(v^{(s)} - v) H_T(v^{(r)} - v)|^{\frac{\alpha}{2}} dv &= \\ &= O \left( \max_{j=\overline{1, n}, s, r=1, L_T, s \neq r} \left( \int_{\Pi} |H_{T_j}(v_j^{(s)} - v_j) H_{T_j}(v_j^{(r)} - v_j)|^{\frac{\alpha}{2}} dv_j \right) \right). \end{aligned}$$

By (29), we obtain the required result.  $\square$

*Proof of Corollary 4.* Since (5) and (4) hold, we have

$$L_T = O(m^{\beta n}), M_T = O(m^{qn}). \quad (40)$$

As it follows from Demesh (1988),

$$\int_{\Pi} \left| H_{T_j}(\nu_j^{(s)} - \nu_j) H_{T_j}(\nu_j^{(r)} - \nu_j) \right|^{\frac{\alpha}{2}} d\nu_j = O\left(m^{-\frac{2k^2\alpha^2(1-\beta)-1}{1+2k\alpha}}\right), \quad (41)$$

$\beta < 1 - \frac{1}{2k^2\alpha^2}$ . Substitute (40) and (41) into (30) to obtain the required result.  $\square$

*Proof of Theorem 3.* We follow the lines of the proof in Masry and Cambanis (1984) for stable random processes.

Let  $f(\lambda^{(0)}) = [\varphi(\lambda^{(0)})]^{\frac{p}{\alpha}}$ . By inequality (32) with  $a = \widehat{f}_T(\lambda^{(0)})$ ,  $b = f(\lambda^{(0)})$ ,  $q = \frac{\alpha}{p}$ , we have

$$\begin{aligned} \left| \widehat{\varphi}_T(\lambda^{(0)}) - \varphi(\lambda^{(0)}) \right| &= \left| \left[ \widehat{f}_T(\lambda^{(0)}) \right]^{\frac{\alpha}{p}} - \left[ f(\lambda^{(0)}) \right]^{\frac{\alpha}{p}} \right| \leq \\ &\leq \frac{\alpha}{2p} \left| \widehat{f}_T(\lambda^{(0)}) - f(\lambda^{(0)}) \right| \left( \left[ \widehat{f}_T(\lambda^{(0)}) \right]^{\frac{\alpha}{p}-1} + \left[ f(\lambda^{(0)}) \right]^{\frac{\alpha}{p}-1} \right). \end{aligned}$$

By the equality

$$E \left| \widehat{f}_T(\lambda^{(0)}) - f(\lambda^{(0)}) \right|^2 = \text{var} \widehat{f}_T(\lambda^{(0)}) + \left( E \widehat{f}_T(\lambda^{(0)}) - f(\lambda^{(0)}) \right)^2, \quad (42)$$

by (19) and (30), we have  $E \left| \widehat{f}_T(\lambda^{(0)}) - f(\lambda^{(0)}) \right|^2 \rightarrow 0$  as  $T \rightarrow \infty$ . Therefore

$$\left[ \widehat{f}_T(\lambda^{(0)}) \right]^{\frac{\alpha}{p}-1} + \left[ f(\lambda^{(0)}) \right]^{\frac{\alpha}{p}-1} \xrightarrow{P} 2 \left[ f(\lambda^{(0)}) \right]^{\frac{\alpha}{p}-1} \text{ as } T \rightarrow \infty.$$

Using the Chebyshev inequality for any  $\epsilon > 0$  we obtain

$$P \left( \left| \widehat{\varphi}_T(\lambda^{(0)}) - \varphi(\lambda^{(0)}) \right| > \epsilon \right) \leq \text{Const} \frac{E \left| \widehat{f}_T(\lambda^{(0)}) - f(\lambda^{(0)}) \right|^2}{\epsilon^2} \rightarrow 0$$

as  $T \rightarrow \infty$ . Theorem 3 is proved.  $\square$

*Proof of Corollary 5.* If  $\gamma \in (0, 1)$ , then by (27), (31) and (42) we obtain

$$E \left| \widehat{f}_T(\lambda^{(0)}) - f(\lambda^{(0)}) \right|^2 = O(m^{-2q\gamma}) + O(m^{-(\beta-q)n}) + O\left(m^{-\frac{2k^2\alpha^2(1-\beta)-1}{1+2k\alpha}}\right).$$

By requiring that all terms on the right-hand side of the last expression tend to zero at the same rate, we obtain

$$\begin{cases} 2q\gamma = (\beta - q)n, \\ (\beta - q)n = \frac{2k\alpha^2(1-\beta)-1}{2+2k\alpha}. \end{cases}$$

Then

$$\beta = \frac{(n+2\gamma)(2k^2\alpha^2-1)}{(2\gamma n(1+2k\alpha)2k^2\alpha^2(n+2\gamma))} \text{ and } (\beta-q)n = \frac{\gamma n(2k^2\alpha^2-1)}{n\gamma(1+2k\alpha)+k^2\alpha^2(n+2\gamma)}.$$

Hence

$$E \left| \widehat{f}_T(\lambda^{(0)}) - f(\lambda^{(0)}) \right|^2 = O \left( m^{-\frac{\gamma n(2k^2\alpha^2-1)}{n\gamma(1+2k\alpha)+k^2\alpha^2(n+2\gamma)}} \right)$$

if  $2k^2\alpha^2 > 1$ . If  $\gamma = 1$ , then by (27), (31) and (42) we obtain

$$\begin{aligned} E \left| \widehat{f}_T(\lambda^{(0)}) - f(\lambda^{(0)}) \right|^2 &= O \left( m^{-2q} \ln^2(m) \right) + O \left( m^{-(\beta-q)n} \ln^2(m) \right) + \\ &+ O \left( m^{-\frac{2k^2\alpha^2(1-\beta)-1}{1+2k\alpha}} \ln^2(m) \right). \end{aligned}$$

By requiring that all items on the right-hand side of the last expression tend to zero at the same rate and making the same calculations as in the case  $\gamma \in (0, 1)$ , we obtain

$$E \left| \widehat{f}_T(\lambda^{(0)}) - f(\lambda^{(0)}) \right|^2 = O \left( m^{-\frac{n(2k^2\alpha^2-1)}{n(1+2k\alpha)+k^2\alpha^2(n+2)}} \ln^2(m) \right)$$

if  $2k^2\alpha^2 > 1$ . By the Chebyshev inequality for a given  $\epsilon > 0$ , we have

$$P \left( a_n \left| \widehat{f}_T(\lambda^{(0)}) - f(\lambda^{(0)}) \right| > \epsilon \right) \leq \frac{a_n^2}{\epsilon^2} E \left| \widehat{f}_T(\lambda^{(0)}) - f(\lambda^{(0)}) \right|^2 \leq \frac{Const}{\epsilon^2} \frac{1}{\ln^2(m)} \rightarrow 0$$

as  $T \rightarrow \infty$ . Corollary 5 is proved.  $\square$

## 5 Appendix 2

**Example 1** Let  $h_T(t) = h(k_j, m_j)$  be a polynomial data window. In order to proof (15), we show that the rate of convergence to zero as  $T \rightarrow 0$  of the integral

$$\int_{\Pi^n \setminus \{\|\lambda\| \leq \delta\}} |H_T(\lambda)|^\alpha d\lambda \quad (43)$$

is greater than that of

$$\int_{\Pi^n} \|\lambda\|^\gamma |H_T(\lambda)|^\alpha d\lambda \quad (44)$$

Let us consider  $H_T(\lambda) = \prod_{j=1}^n H_{T_j}(\lambda_j)$ ,  $\lambda \in \Pi^n$ . Then integral (43) is a product of integrals of the form  $\int_{\Pi \setminus \{|\lambda_j| \leq \delta\}} |H_{T_j}(\lambda_j)|^\alpha d\lambda_j$  for all  $j = \overline{1, n}$ . It follows from



Demesh (1988) that for any  $0 < \delta < \pi$  fixed and for any  $j$ ,  $1 \leq j \leq n$ , we have  $\int_{\Pi \setminus \{|\lambda_j| \leq \delta\}} |H_{T_j}(\lambda_j)|^\alpha d\lambda_j = O\left(\frac{1}{\tau_j^{2k_j\alpha-1}}\right)$  as  $\tau_j \rightarrow \infty$  for polynomial data windows satisfying (26). The last expression tends to zero if  $2k_j\alpha - 1 > 0$ .

By (34),  $\int_{\Pi^n} \|\mu\|^\gamma |H_T(\mu)|^\alpha d\mu \leq 2^n n \max_{j=1, \dots, n} \left( \int_{\Pi} |\mu_j|^\gamma |H_{T_j}(\mu_j)|^\alpha d\mu_j \right)$ . For any  $j$ ,  $1 \leq j \leq n$ ,  $\int_{\Pi} |\mu_j|^\gamma |H_{T_j}(\lambda_j)|^\alpha d\lambda_j = O\left(\frac{1}{\tau_j^\gamma}\right)$  as  $\tau_j \rightarrow \infty$  (see Demesh (1988)). Therefore if  $k_j > \frac{\gamma+1}{2\alpha}$ , the integral  $\int_{\Pi \setminus \{|\lambda_j| \leq \delta\}} |H_{T_j}(\lambda_j)|^\alpha d\lambda_j$  tends to zero faster than the integral  $\int_{\Pi} |\mu_j|^\gamma |H_{T_j}(\lambda_j)|^\alpha d\lambda_j$ . Since  $k_j \in \left\{\frac{1}{2}\right\} \cup \mathbb{N}$  and  $\gamma \in (0, 1]$ , the condition  $2k_j\alpha - 1 > 0$  hold. Hence integral (43) tends to zero faster than integral (44).

**Example 2** Let  $h_T(t) = h(k_j, m_j)$  be a polynomial data window. Consider  $H_T(\lambda) = \prod_{j=1}^n H_{T_j}(\lambda_j)$ . Then integral (28) is a product of one-dimensional integrals of the form

$$\max_{s,r=1, L_{T_j}, s \neq r} \int_{\Pi} \left| H_{T_j}(\nu_j^{(s)} - \nu_j) H_{T_j}(\nu_j^{(r)} - \nu_j) \right|^{\frac{\alpha}{2}} d\nu_j. \quad (45)$$

As it follows from Demesh (1988), the following condition holds for all  $j$ ,  $1 \leq j \leq n$ :

$$\max_{s,r=1, L_{T_j}, s \neq r} \int_{\Pi} \left| H_{T_j}(\nu_j^{(s)} - \nu_j) H_{T_j}(\nu_j^{(r)} - \nu_j) \right|^{\frac{\alpha}{2}} d\nu_j = O\left(\tau_j^{-\frac{2k_j^2\alpha^2-1}{2k_j\alpha+1}}\right) \text{ as } \tau_j \rightarrow \infty \quad (46)$$

if  $2k_j^2\alpha^2 - 1 > 0$ . Since each of the integrals of form (45) tends to zero, their product also tends to zero.

**Example 3** Let  $W_T(\nu) = \prod_{i=1}^n W_{T_i}(\nu_i)$  where  $W_{T_i}(\nu_i)$ ,  $\nu_i \in \Pi$ , satisfy (35),  $\gamma \in (0, 1)$ . Then the integral  $\int_{\Pi^n \setminus \{\|\nu\| \leq \delta\}} W_T(\nu) d\nu$  is a product of integrals of the form  $\int_{\Pi \setminus \{|\nu_j| \leq \delta\}} W_{T_j}(\nu_j) d\nu_j$ . As it follows from Trush (1999), we have

$$\int_{\Pi \setminus \{|\nu_j| \leq \delta\}} W_{T_j}(\nu_j) d\nu_j = O\left(\frac{1}{M_{T_j}^\gamma}\right) \text{ as } T \rightarrow \infty. \quad (47)$$

By (33) and (36), we obtain

$$\int_{\Pi^n} \|\nu\|^\gamma W_T(\nu) d\nu = O\left(\frac{1}{M_{T_j}^\gamma}\right) \text{ as } T \rightarrow \infty \quad \text{if } \gamma \in (0, 1) \quad (48)$$

Since  $\gamma \in (0, 1)$ , integral (47) tends to zero faster than integral (48).

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